

CALCULUS II: TUTORIAL (2) SOLUTIONS

① $32\sqrt{2}(1-i) = 64 \left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} \right) = 64 \left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right) \right]$

By De Moivre's thm, one sixth root is

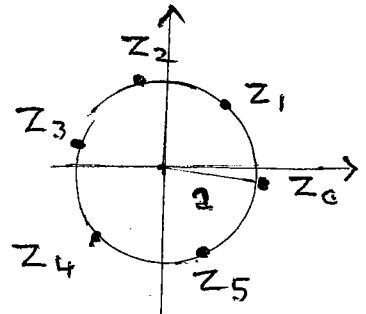
$$Z_0 = 64^{\frac{1}{6}} \left[\cos\left(-\frac{\pi}{24}\right) + i\sin\left(-\frac{\pi}{24}\right) \right] \quad (64^{\frac{1}{6}} = 2)$$

Since the roots are equally spaced around the circle $r=2$ with separation $\frac{2\pi}{6} = 60^\circ$, the others are

$$Z_1 = 2 \operatorname{cis} \left(-\frac{\pi}{24} + \frac{\pi}{3} \right) = 2 \operatorname{cis} \left(\frac{7\pi}{24} \right)$$

$$Z_2 = 2 \operatorname{cis} \left(\frac{15\pi}{24} \right), \quad Z_3 = 2 \operatorname{cis} \left(\frac{23\pi}{24} \right)$$

$$Z_4 = 2 \operatorname{cis} \left(\frac{31\pi}{24} \right), \quad Z_5 = 2 \operatorname{cis} \left(\frac{39\pi}{24} \right)$$



The next root $2 \operatorname{cis} \left(\frac{47\pi}{24} \right)$ is the same as Z_0

②
$$\frac{(\sqrt{3}+i)^2(1-i)^3}{(i\sqrt{3}-1)^3(i+1)^2} = \frac{[2(\cos 30^\circ + i\sin 30^\circ)]^2 [\sqrt{2}(\cos 45^\circ - i\sin 45^\circ)]^3}{[2(\cos 120^\circ + i\sin 120^\circ)]^3 [\sqrt{2}(\cos 45^\circ + i\sin 45^\circ)]^2}$$

$$= \frac{4(\cos 60^\circ + i\sin 60^\circ) 2\sqrt{2}(\cos 135^\circ - i\sin 135^\circ)}{8(\cos 360^\circ + i\sin 360^\circ) 2(\cos 90^\circ + i\sin 90^\circ)}$$

$$= \frac{\sqrt{2}}{2} [\cos(60^\circ - 135^\circ - 360^\circ - 90^\circ) + i\sin(\quad)]$$

$$= \frac{1}{\sqrt{2}} [\cos(-525^\circ) + i\sin(-525^\circ)] \quad (\text{add } 4\pi = 720^\circ)$$

$$= \frac{1}{\sqrt{2}} (\cos 195^\circ + i\sin 195^\circ) \Rightarrow r = \frac{1}{\sqrt{2}}, \quad \theta = 195^\circ = \frac{13\pi}{12}$$

Another method: $i\sqrt{3}-1 = i(\sqrt{3}+i)$, $i+1 = i(1-i)$

so expression becomes $\frac{(1-i)}{i^5(\sqrt{3}+i)} = -i \frac{(1-i)(\sqrt{3}-i)}{(\sqrt{3}+i)(\sqrt{3}-i)} =$

$$= -\left(\frac{\sqrt{3}+1}{4}\right) - i\left(\frac{\sqrt{3}-1}{4}\right) = \frac{1}{\sqrt{2}} \left[-\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right) - i\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right) \right]$$

$$\cos 15^\circ = \cos(45^\circ - 30^\circ) = \frac{1}{\sqrt{2}} \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}} = -\cos 195^\circ$$

$$\sin 15^\circ = \sin(45^\circ - 30^\circ) = \frac{1}{\sqrt{2}} \frac{1}{2} - \frac{1}{\sqrt{2}} \frac{\sqrt{3}}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}} = -\sin 195^\circ$$

This is more difficult but instructive

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$$\begin{aligned}
 & \frac{(\cos 3\theta + i \sin 3\theta)^6 (\cos 2\theta - i \sin 2\theta)^7}{(\cos 5\theta + i \sin 5\theta)^6 (\cos \theta - i \sin \theta)^8} \quad (\text{use de Moivre}) \\
 &= \frac{(\cos \theta + i \sin \theta)^{18} (\cos \theta - i \sin \theta)^{14}}{(\cos \theta + i \sin \theta)^{30} (\cos \theta - i \sin \theta)^8} \quad [i^6 = i^2 \cdot i^4 = -1] \\
 &= (\cos \theta + i \sin \theta)^{18-30} (\cos \theta - i \sin \theta)^{14-8} \quad [\text{cis}(-\theta) = (\text{cis} \theta)^{-1}] \\
 &= (\cos \theta + i \sin \theta)^{-12} (\cos \theta + i \sin \theta)^6 \\
 &= (\cos \theta + i \sin \theta)^{-18} = \cos(18\theta) - i \sin(18\theta)
 \end{aligned}$$

Note: quicker to use $e^{i\theta}$ or $\text{cis} \theta$ notation

$$\begin{aligned}
 4) \quad \cos 4\theta + i \sin 4\theta &= (\cos \theta + i \sin \theta)^4 + (i \sin \theta)^4 \\
 &= \cos^4 \theta + 4 \cos^3 \theta (i \sin \theta) + 6 \cos^2 \theta (i \sin \theta)^2 + 4 \cos \theta (i \sin \theta)^3 + (i \sin \theta)^4 \\
 &= \{\cos^4 \theta - 6 \cos^2 \theta \sin^2 \theta + \sin^4 \theta\} + i \{\cos^3 \theta \sin \theta - 4 \cos \theta \sin^3 \theta\} \\
 &\text{Equating real and imaginary parts gives } \cos 4\theta, \sin 4\theta
 \end{aligned}$$

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$$\begin{aligned}
 y &= \sinh x = \frac{1}{2}(e^x + e^{-x}) \Rightarrow e^x - 2y - e^{-x} = 0 \\
 \Rightarrow e^{2x} - 2ye^x - 1 &= 0 \quad (\text{quadratic in } e^x) \\
 \Rightarrow e^x &= y \pm \sqrt{y^2 + 1} \quad (\text{only } + \text{ root permitted}) \\
 \Rightarrow x &= \sinh^{-1} y = \ln(y + \sqrt{y^2 + 1})
 \end{aligned}$$

$$\begin{aligned}
 \sinh z &= \sinh(x + iy) = \sinh x \cosh(iy) + \cosh x \sinh(iy) \\
 &= \sinh x \cos y + i \cosh x \sin y
 \end{aligned}$$

$$\Rightarrow -2 = \sinh x \cos y + i \cosh x \sin y$$

Equating real and imaginary parts:

$$0 = \cosh x \sin y \Rightarrow \sin y = 0 \quad (\cosh x \neq 0) \Rightarrow y = n\pi \quad \begin{matrix} \text{integer} \\ \downarrow \end{matrix}$$

$$-2 = \sinh x \cos y = (-1)^n \sinh x$$

$$\Rightarrow x = \sinh^{-1} [2(-1)^{n+1}] = \ln [2(-1)^{n+1} + \sqrt{4+1}]$$

$$\Rightarrow z = \ln [2(-1)^{n+1} + \sqrt{5}] + i n \pi$$